

Investigation of PT -symmetric Hamiltonian systems from an alternative point of view

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Abstract

Two non-Hermitian PT -symmetric Hamiltonian systems are reconsidered by means of the algebraic method which was originally proposed for the pseudo-Hermitian Hamiltonian systems rather than for the PT -symmetric ones. Compared with the way converting a non-Hermitian Hamiltonian to its Hermitian counterpart, this method has the merit that keeps the Hilbert space of the non-Hermitian PT -symmetric Hamiltonian unchanged. In order to give the positive definite inner product for the PT -symmetric systems, a new operator V , instead of C , can be introduced. The operator V has the similar function to the operator C adopted normally in the PT -symmetric quantum mechanics, however, it can be constructed, as an advantage, directly in terms of Hamiltonians. The spectra of the two non-Hermitian PT -symmetric systems are obtained, which coincide with that given in literature, and in particular, the Hilbert spaces associated with positive definite inner products are worked out.

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1 Introduction

One class of non-Hermitian Hamiltonians has positive and real spectra [1, 2] if the non-Hermitian Hamiltonian satisfies the condition: both the Hamiltonian and its eigenfunctions are PT invariant, where the linear parity operator P reverses the position and momentum: $x \rightarrow -x$, $p \rightarrow -p$, and the antilinear time reversal operator T reverses the momentum and imaginary unit: $p \rightarrow -p$, $i \rightarrow -i$. If the inner product of two states $\varphi(x)$ and $\phi(x)$ is defined [3] to be: $\langle \varphi(x), \phi(x) \rangle_{PT} \equiv \int [PT\varphi(x)]\phi(x)dx$, such a quantity is not positive definite. This problem has been overcome [4] by introducing the operator C that commutes with both the non-Hermitian PT -symmetric Hamiltonian and the combined operator PT . That is to say, the CPT inner product turns out to be positive definite, the Hamiltonian and its transposition are related [4] by the CPT similarity transformation in addition to the Hamiltonian's PT symmetry, and the time evolution generated by the Hamiltonian is kept unitary in the PT -symmetric theory. Therefore, the eigenfunctions of the non-Hermitian PT -symmetric Hamiltonian can be orthogonal and complete [5] such that the non-Hermitian PT -symmetric theory can have a probability interpretation. In addition, the breaking of the PT symmetry has been observed in experiments [6] in the realm of optics. Consequently, the basic frame of the non-Hermitian PT -symmetric quantum mechanics has been established.

Another class of non-Hermitian quantum theory that has been studied recently is closely related to a pseudo-Hermitian (or quasi-Hermitian) Hamiltonian [7, 8, 9, 10, 11]. The pseudo-Hermitian theory with an indefinite metric operator η was first proposed by Pauli [7] in 1943 for the sake of overcoming the divergence of quantum field theories. Later, the theory with a positive definite metric η_+ was developed by others [8, 9, 10, 11]. That is, the Hamiltonian is η_+ pseudo-Hermitian self-adjoint and its eigenfunctions have positive definite inner products with respect to this positive definite metric. As an important progress of the two classes of the non-Hermitian quantum mechanics, there is an intimate relation [12] between the PT -symmetric and the pseudo-Hermitian Hamiltonians, i.e., an exact antilinear symmetric system¹ can be transformed into its corresponding Hermitian system through a similarity transformation, where the similarity transformation can be realized by means of the indefinite metric operator η . Specifically, a PT -symmetric Hamiltonian can correspond to a Hermitian one. The method of transforming a non-Hermitian system into a Hermitian one has frequently been used recently, such as dealing with the fourth-order derivative Pais-Uhlenbeck oscillator model [13] and the PT -symmetric Hamiltonian systems that are composed of interacting non-Hermitian and Hermitian Hamiltonians [14].

However, we have to point out that the method used in ref. [14], i.e., converting a non-Hermitian PT -symmetric model to its corresponding Hermitian one by means of a similarity transformation, alters the Hilbert space of the non-Hermitian Hamiltonian system. In other

¹The PT symmetry is only a specific case of the antilinear symmetry.

words, one can easily verify that the commutator of the two Hamiltonians (the non-Hermitian Hamiltonian and its Hermitian counterpart) is non-vanishing, which means that they give rise to different Hilbert spaces, and that the similarity transformation only ensures the same spectrum for the two Hamiltonians. To this end, we need an improved method which is available for one to get the spectrum of the non-Hermitian PT -symmetric Hamiltonian but not to alter the Hilbert space spanned by the eigenfunctions of the Hamiltonian. Actually, such a method, called the algebraic method, has already been proposed [15] for the pseudo-Hermitian Hamiltonian systems [7, 8, 9, 10, 11]. Here we find that this algebraic method is also available to the non-Hermitian PT -symmetric Hamiltonian systems analyzed in detail in ref. [14]. That is, we shall investigate in terms of the algebraic method the two non-Hermitian PT -symmetric Hamiltonian systems considered in ref. [14], and achieve the goal that the same spectra as that given in ref. [14] are obtained and further the Hilbert spaces with positive definite inner products are worked out. We note that it is a key step to construct the operator V that is model dependent like the operator C . The operator V has the similar function to the operator C adopted [4] normally in the PT -symmetric quantum mechanics, that is, to make the inner product positive definite. The reason for us to introduce V is that it can be constructed directly in terms of the Hamiltonians of quantum systems, and thus the formulation of V is more intuitive than that of C .

This paper is organized as follows. In the next section, we first diagonalize the non-Hermitian PT -symmetric Hamiltonian [14] which is composed of two coupled PT -symmetric Hamiltonians, where one is Hermitian and the other non-Hermitian. Then, we construct the operator V which ensures that the PT -symmetric system now has a positive definite inner product with respect to the combined operator PTV . Next, we redefine the annihilation and creation operators for the PT -symmetric Hamiltonian and give the real spectrum with a lower bound by means of the algebraic method. The spectrum we obtain is the same as that given in ref. [14], and in particular, we provide the complete set of eigenfunctions with the positive definite PTV inner product. In section 3, we extend our investigation to a more complicated model composed of two coupled non-Hermitian PT symmetric Hamiltonians and fulfill the task similar to that of section 2. Finally, section 4 is devoted to a brief conclusion.

2 Model 1: A coupled Hermitian and non-Hermitian PT -symmetric Hamiltonian system

We deal with the model given in ref. [14] by means of the algebraic method [15]. The Hamiltonian of the model takes the form,

$$H = (p_1^2 + x_1^2) + (p_2^2 + x_2^2 + i2x_2) + 2\epsilon x_1 x_2, \quad (1)$$

which is composed of the (Hermitian) harmonic oscillator Hamiltonian, the non-Hermitian PT -symmetric Hamiltonian and the interacting Hamiltonian with the coupling constant² ϵ . Note that (x_j, p_j) , where $j = 1, 2$, are two pairs of canonical variables that satisfy the usual commutation relations,

$$[x_j, p_k] = i\delta_{jk}, \quad [x_j, x_k] = 0 = [p_j, p_k], \quad j, k = 1, 2, \quad (2)$$

where \hbar is set to be unity throughout this paper.

As pointed out in the above section, the Hamiltonian eq. (1) was converted to the following Hermitian one by a similarity transformation in ref. [14],

$$\mathbf{H} = (p_1^2 + x_1^2) + (p_2^2 + x_2^2) + 2\epsilon x_1 x_2 + \frac{1}{1 - \epsilon^2}. \quad (3)$$

It is easy to check that $[H, \mathbf{H}] = -4p_2 \neq 0$, which means that H and \mathbf{H} have different sets of eigenfunctions, i.e., they give different Hilbert spaces. Thus H and \mathbf{H} describe two different systems although they have the same spectrum. In order to find out the Hilbert space with the positive definite inner products for the non-Hermitian system H , we turn to the use of the algebraic method [15] which has been proved to be available in dealing with the pseudo-Hermitian systems.

2.1 Diagonalization

Let us diagonalize the Hamiltonian eq. (1). Applying the way used in ref. [14] directly to this non-Hermitian Hamiltonian, we introduce the new variables of phase space (X_j, P_j) , where $j = 1, 2$, and establish the relations between the new and original variables as follows,

$$p_1 = aP_1 + bP_2, \quad p_2 = cP_1 + dP_2, \quad x_1 = eX_1 + fX_2, \quad x_2 = gX_1 + hX_2, \quad (4)$$

where a, b, c, d, e, f, g , and h are unknown real coefficients. Furthermore, we impose the canonical commutation relations as eq. (2) to the new variables,

$$[X_j, P_k] = i\delta_{jk}, \quad [X_j, X_k] = 0 = [P_j, P_k], \quad j, k = 1, 2. \quad (5)$$

We determine the unknown coefficients by requiring that (i) The canonical commutation relations for the new variables and for the original ones should be consistent to each other, and (ii) The cross terms of the Hamiltonian expressed in terms of the new variables (X_j, P_j)

²The energy spectrum is real and positive when $|\epsilon| < 1$. The critical value is at $|\epsilon| = 1$, and the spectrum becomes complex when $|\epsilon| > 1$. For the details, see ref. [14]. In the present paper we focus first on the region of $|\epsilon| < 1$, and then point out particularly that the reason that the complex spectrum occurs if $|\epsilon| > 1$ is just the breaking of the PT symmetry, which was unanswered in ref. [14] because the eigenfunctions of the Hamiltonian system were not obtained there.

should be vanished when eq. (4) is substituted into eq. (1). The two sets of conditions give six equations for the eight unknown coefficients, and the solutions are

$$c = \zeta a, \quad d = -\zeta b, \quad e = \frac{1}{2a}, \quad f = \frac{1}{2b}, \quad g = \frac{\zeta}{2a}, \quad h = -\frac{\zeta}{2b}, \quad (6)$$

where $\zeta = \pm 1$, and a and b are arbitrary non-vanishing real parameters. As a result, the Hamiltonian eq. (1) is now diagonalized as

$$H = 2a^2 P_1^2 + \frac{1 + \zeta\epsilon}{2a^2} \left(X_1 + i \frac{a\zeta}{1 + \zeta\epsilon} \right)^2 + 2b^2 P_2^2 + \frac{1 - \zeta\epsilon}{2b^2} \left(X_2 - i \frac{b\zeta}{1 - \zeta\epsilon} \right)^2 + \frac{1}{1 - \epsilon^2}, \quad (7)$$

and the relations between the new and original variables are given by

$$\begin{aligned} P_1 &= \frac{1}{2a} (p_1 + \zeta p_2), & P_2 &= \frac{1}{2b} (p_1 - \zeta p_2), \\ X_1 &= a (x_1 + \zeta x_2), & X_2 &= b (x_1 - \zeta x_2). \end{aligned} \quad (8)$$

In order to utilize the algebraic method conveniently, we further introduce two pairs of variables in phase space, $(\mathcal{X}_j, \mathcal{P}_j)$, where $j = 1, 2$, and rewrite the above Hamiltonian (eq. (7)) in a completely diagonalized form,

$$\begin{aligned} H &= H_1 + H_2 + \frac{1}{1 - \epsilon^2}, \\ H_1 &= 2a^2 \mathcal{P}_1^2 + \frac{1 + \zeta\epsilon}{2a^2} \mathcal{X}_1^2, \\ H_2 &= 2b^2 \mathcal{P}_2^2 + \frac{1 - \zeta\epsilon}{2b^2} \mathcal{X}_2^2, \end{aligned} \quad (9)$$

where \mathcal{X}_j and \mathcal{P}_j are defined as follows:

$$\begin{aligned} \mathcal{P}_1 &:= P_1, & \mathcal{P}_2 &:= P_2, \\ \mathcal{X}_1 &:= X_1 + i \frac{a\zeta}{1 + \zeta\epsilon}, & \mathcal{X}_2 &:= X_2 - i \frac{b\zeta}{1 - \zeta\epsilon}. \end{aligned} \quad (10)$$

We emphasize that \mathcal{P}_j 's are still Hermitian while \mathcal{X}_j 's non-Hermitian due to the non-Hermiticity of the Hamiltonian eq. (1), which is different from the case occurred in ref. [14] but suitable for being dealt with by the algebraic method [15]. Note that $(\mathcal{X}_j, \mathcal{P}_j)$ satisfy the same commutation relations as (X_j, P_j) ,

$$[\mathcal{X}_j, \mathcal{P}_k] = i\delta_{jk}, \quad [\mathcal{X}_j, \mathcal{X}_k] = 0 = [\mathcal{P}_j, \mathcal{P}_k], \quad j, k = 1, 2, \quad (11)$$

which meet the basic requirement for us to apply the algebraic method to the Hamiltonian system described by eqs. (9) and (10).

2.2 CPT inner product and its shortcoming

We deviate our goal temporarily and mention the normally used CPT inner product and its shortcoming, which may provide some reason for us to adopt our PTV inner product (or its equivalent PV -pseudo inner product) in the next subsection. As analyzed in refs. [4, 5], the inner product of eigenfunctions in PT -symmetric Hamiltonian systems is not positive definite,

$$\langle \varphi_n(x), \varphi_m(x) \rangle_{PT} \equiv \int [PT\varphi_n(x)]\varphi_m(x)dx = (-1)^n \delta_{nm}, \quad (12)$$

where $\{\varphi_n(x), n \in \mathbb{N}\}$ is the set of eigenfunctions of a PT -symmetric Hamiltonian. In order to overcome this difficulty, a linear operator C is constructed [4] in terms of the set of eigenfunctions in such a way that it commutes with both the PT -symmetric Hamiltonian and the combined operator PT , and in particular that it has the following desired property,

$$C\varphi_n(x) = (-1)^n \varphi_n(x), \quad (13)$$

where the property $C^2 = 1$ is obvious. Consequently, the CPT inner product turns out to be positive definite,

$$\langle \varphi_n(x), \varphi_m(x) \rangle_{CPT} \equiv \int [CPT\varphi_n(x)]\varphi_m(x)dx = \delta_{nm}. \quad (14)$$

However, the operator C is unknown before the eigenfunctions of a non-Hermitian PT -symmetric Hamiltonian system are solved, and it is hard to be expressed concisely even after the eigenfunctions are obtained. This reminds us to search for an alternative operator which, associated directly with the non-Hermitian PT -symmetric Hamiltonian rather than its eigenfunctions, not only maintains the desired property (eq. (13)) but also is easy to be constructed. Fortunately, such a substitutor can be found out. See the definition of the operator V in the next subsection.

2.3 PTV inner product and its advantage

Now we turn to our PTV inner product (or its equivalent PV -pseudo inner product) in this subsection and then give in the next subsection the energy spectrum and eigenfunctions for the non-Hermitian PT -symmetric system described by the Hamiltonian eq. (1) or eq. (9).

According to the PT -symmetric quantum mechanics with the positive definite CPT inner product [4], a Hamiltonian \mathcal{H} , in addition to the PT symmetry $\mathcal{H} = (PT)^{-1}\mathcal{H}(PT)$, is required to satisfy

$$\mathcal{H} = (CPT)^{-1}\tilde{\mathcal{H}}(CPT), \quad (15)$$

where the tilde stands for transposition. Considering the properties $[P, T] = 0$ and $[C, PT] = 0$ given in ref. [4], we can reduce eq. (15) to

$$\mathcal{H} = (TPC)^{-1}\tilde{\mathcal{H}}(TPC) = (PC)^{-1} \cdot T^{-1}\tilde{\mathcal{H}}T \cdot (PC) = (PC)^{-1}\mathcal{H}^\dagger(PC), \quad (16)$$

where the dagger means Hermitian conjugate. We note that eq. (16) establishes a relationship between a non-Hermitian PT -symmetric quantum system and a PC pseudo-Hermitian one [7]. That is, the requirement that a PT -symmetric Hamiltonian has a positive definite CPT inner product leads to the result³ that this PT -symmetric Hamiltonian must be PC pseudo-Hermitian self-adjoint. Consequently, we can bring the PT -symmetric Hamiltonian system into the framework of the PC pseudo-Hermitian Hamiltonian system. Different from ref. [11], our proof is achieved at the level of Hamiltonians, which is not involved in eigenfunctions that are hard to be solved sometimes (see also footnote 3).

The key step of the algebraic method is to work out the operator V that has the similar function to that of C but is only associated with the Hamiltonian of the system we are investigating, while the operator C is relevant to the eigenfunctions of the Hamiltonian. The operator V should be constructed [15] in terms of the Hamiltonian (see eqs. (9) and (10)) in the following way,

$$V = e^{i\pi\left(\frac{H_1}{2\sqrt{1+\zeta\epsilon}} + \frac{H_2}{2\sqrt{1-\zeta\epsilon}} - 1\right)}. \quad (17)$$

On the one hand, from the point of view of the PT -symmetric quantum mechanics, one can verify that V indeed has the same properties⁴ as C . That is, V is non-Hermitian but PT -symmetric, and it commutes with both PT and the Hamiltonian (eq. (1) or eq. (9)). On the other hand, from the point of view of the pseudo-Hermitian quantum mechanics, one can verify that V is P -pseudo-Hermitian self-adjoint because the Hamiltonian (eq. (1) or eq. (9)) is P -pseudo-Hermitian self-adjoint, i.e., $V = P^{-1}V^\dagger P$ due to $H = P^{-1}H^\dagger P$, the positive definite metric operator defined by $\eta_+ := PV$ is Hermitian as desired, i.e., $\eta_+^\dagger = V^\dagger P^\dagger = V^\dagger P = P(P^{-1}V^\dagger P) = PV = \eta_+$, and therefore the Hamiltonian (eq. (1) or eq. (9)) is further η_+ -pseudo-Hermitian self-adjoint, i.e., $H = \eta_+^{-1}H^\dagger \eta_+ = (PV)^{-1}H^\dagger(PV)$. In particular, the pseudo inner product associated with the positive definite metric operator η_+ can be shown⁵ to be equivalent to the PTV inner product and thus it is positive definite. The advantage of using V instead of C is not only that V can easily be constructed but also that we are enlightened to apply the algebraic method (that was originally proposed for the pseudo-Hermitian quantum mechanics) to the PT -symmetric quantum mechanics.

³Although it has been mentioned in ref. [11], this result is obtained here with no use of the postulation that \mathcal{H} has a complete biorthonormal eigenbasis and a discrete spectrum. We note that this postulation that is not mandatory here is crucial for the outcomes deduced in ref. [11].

⁴The property $V^2 = 1$ will be shown after V 's eigenfunctions that are also the eigenfunctions of the Hamiltonian (eq. (1) or eq. (9)) are solved.

⁵In the PT -symmetric quantum mechanics, the PTV inner product is defined as $\langle\varphi(x), \phi(x)\rangle_{PTV} \equiv \int [PTV\varphi(x)]\phi(x)dx$, while in the PV -pseudo-Hermitian quantum mechanics, the PV -pseudo inner product is defined as $\langle\varphi(x)|\phi(x)\rangle_{PV} \equiv \int \overline{\varphi(x)}PV\phi(x)dx$, where the overline stands for complex conjugate. By using the properties: $[P, T] = 0$, $[V, PT] = 0$, and $(PV)^\dagger = PV$, one can prove that the two definitions of inner products are equivalent, i.e., $\langle\varphi(x), \phi(x)\rangle_{PTV} = \langle\varphi(x)|\phi(x)\rangle_{PV}$.

By means of the operator V given above, we can now define new annihilation and creation operators by following the algebraic method. When the annihilation operators take the forms,

$$\begin{aligned} a_1 &= \frac{a}{\sqrt[4]{1+\zeta\epsilon}} \left(i\mathcal{P}_1 + \frac{\sqrt{1+\zeta\epsilon}}{2a^2} \mathcal{X}_1 \right), \\ a_2 &= \frac{b}{\sqrt[4]{1-\zeta\epsilon}} \left(i\mathcal{P}_2 + \frac{\sqrt{1-\zeta\epsilon}}{2b^2} \mathcal{X}_2 \right), \end{aligned} \quad (18)$$

the corresponding creation operators are defined as the PV -pseudo Hermitian adjoint of the annihilation operators,⁶

$$\begin{aligned} a_1^\dagger &\equiv (PV)^{-1} a_1^\dagger (PV) = \frac{a}{\sqrt[4]{1+\zeta\epsilon}} \left(-i\mathcal{P}_1 + \frac{\sqrt{1+\zeta\epsilon}}{2a^2} \mathcal{X}_1 \right), \\ a_2^\dagger &\equiv (PV)^{-1} a_2^\dagger (PV) = \frac{b}{\sqrt[4]{1-\zeta\epsilon}} \left(-i\mathcal{P}_2 + \frac{\sqrt{1-\zeta\epsilon}}{2b^2} \mathcal{X}_2 \right). \end{aligned} \quad (19)$$

Using the commutation relations eq. (11) satisfied by the variables of phase space $(\mathcal{X}_j, \mathcal{P}_j)$, where $j = 1, 2$, we can verify that the newly defined annihilation and creation operators satisfy the expected algebraic relations

$$[a_j, a_k^\dagger] = \delta_{jk}, \quad [a_j, a_k] = 0 = [a_j^\dagger, a_k^\dagger], \quad j, k = 1, 2. \quad (20)$$

We define the number operator⁷ in the similar way to that in the conventional quantum mechanics,

$$N_j = a_j^\dagger a_j, \quad j = 1, 2, \quad (21)$$

and obtain the other expected algebraic relations by using eqs. (20) and (21),

$$[N_j, a_k^\dagger] = a_j^\dagger \delta_{jk}, \quad [N_j, a_k] = -a_j \delta_{jk}, \quad j, k = 1, 2. \quad (22)$$

Furthermore, for a given set of eigenstates of the number operator N_j , i.e., $|n_j\rangle$, we have

$$N_j |n_j\rangle = n_j |n_j\rangle, \quad j = 1, 2. \quad (23)$$

When considering the equivalence between the PTV inner product in the PT -symmetric quantum mechanics and the PV -pseudo inner product in the PV -pseudo Hermitian quantum, and utilizing the positive definiteness of the two classes of inner products (see eq. (14) and footnote 5), we finally convince that the operators a_j and a_j^\dagger (see eqs. (18) and (19))

⁶Within the framework of the PT -symmetric quantum mechanics, the creation operators can also be expressed as $a_1^\dagger = (PTV)^{-1} \tilde{a}_1 (PTV)$ and $a_2^\dagger = (PTV)^{-1} \tilde{a}_2 (PTV)$. We can verify them easily by referring to eq. (16).

⁷Repeated subscripts do not sum except for extra indications.

are indeed annihilation and creation operators, respectively, and have the property of ladder operators,

$$a_j^\dagger |n_j\rangle = \sqrt{n_j + 1} |n_j + 1\rangle, \quad a_j |n_j\rangle = \sqrt{n_j} |n_j - 1\rangle, \quad j = 1, 2. \quad (24)$$

Consequently, we rewrite the PT -symmetric Hamiltonian (see eqs. (9) and (10)) in terms of the number operators as follows:

$$H = \sqrt{1 + \zeta\epsilon} (2N_1 + 1) + \sqrt{1 - \zeta\epsilon} (2N_2 + 1) + \frac{1}{1 - \epsilon^2}. \quad (25)$$

2.4 Spectrum and eigenfunction

Now we obtain the energy spectrum from eq. (25),

$$E_{n_1 n_2} = \sqrt{1 + \zeta\epsilon} (2n_1 + 1) + \sqrt{1 - \zeta\epsilon} (2n_2 + 1) + \frac{1}{1 - \epsilon^2}, \quad (26)$$

where $n_1, n_2 \in \mathbb{N}$. This result is obviously same as that given in ref. [14] but here it is derived in terms of the algebraic method, which shows that the algebraic method is also available for non-Hermitian PT -symmetric quantum systems.

Next, we focus on the eigenfunctions of the system described by the Hamiltonian eq. (9) or eq. (25), which is beyond the context of ref. [14]. Quite similar to the case of the two separate harmonic oscillators, we solve $H\varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2) = E_{n_1 n_2} \varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2)$ and obtain the eigenfunctions with the help of the Mathematica,

$$\varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2) = \varphi_{n_1}(\mathcal{X}_1) \varphi_{n_2}(\mathcal{X}_2), \quad (27)$$

where \mathcal{X}_1 and \mathcal{X}_2 are now denoted as the coordinates whose operators are defined in eq. (10), and the eigenfunctions of the “single harmonic oscillator” take the form,

$$\varphi_{n_j}(\mathcal{X}_j) = \frac{\sqrt{c_j}}{\sqrt[4]{\pi}} (2^{n_j} n_j!)^{-\frac{1}{2}} e^{-\frac{1}{2}(c_j \mathcal{X}_j)^2} H_{n_j}(c_j \mathcal{X}_j), \quad j = 1, 2. \quad (28)$$

Note that $H_{n_j}(c_j \mathcal{X}_j)$ is the Hermite polynomial of the n_j -th degree, where c_j 's are parameters given by

$$c_1 = \frac{\sqrt[4]{1 + \zeta\epsilon}}{\sqrt{2a^2}}, \quad c_2 = \frac{\sqrt[4]{1 - \zeta\epsilon}}{\sqrt{2b^2}}, \quad (29)$$

which are real when $|\epsilon| < 1$.

At this stage we can complete the proof of the property $V^2 = 1$ for the operator V (see eq. (17)) and the positive definiteness of inner products. As V commutes with the Hamiltonian (see eq. (9) or eq. (25)), $\varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2)$ is also the set of eigenfunctions of V . Therefore, by using eqs. (17), (27), and (28) we get

$$V\varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2) = (-1)^{n_1 + n_2} \varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2), \quad (30)$$

which gives rise to the expected property $V^2 = 1$. Furthermore, the above equation coincides with that of the operator C (see eq. (13)) used for constructing the positive definite inner product in PT -symmetric systems. As analyzed in ref. [15], by using the Cauchy's residue theorem, the properties of the Hermite polynomials and eqs. (27)-(30), we can verify that the PV -pseudo inner product of the eigenfunctions is positive definite and orthogonal⁸,

$$\begin{aligned}
& \langle \varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2) | \varphi_{m_1 m_2}(\mathcal{X}_1, \mathcal{X}_2) \rangle_{PV} \\
& \equiv \int_{-\infty+iI_1}^{+\infty+iI_1} \int_{-\infty+iI_2}^{+\infty+iI_2} \bar{\varphi}_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2) PV \varphi_{m_1 m_2}(\mathcal{X}_1, \mathcal{X}_2) d\mathcal{X}_1 d\mathcal{X}_2 \\
& = \delta_{n_1 m_1} \delta_{n_2 m_2},
\end{aligned} \tag{31}$$

where I_1 and I_2 are two real parameters which can be determined from eq. (10), i.e., $I_1 = \frac{a\zeta}{1+\zeta\epsilon}$ and $I_2 = -\frac{b\zeta}{1-\zeta\epsilon}$.

2.5 Breaking of the PT symmetry

In the above subsections we focus only on the region of $|\epsilon| < 1$ in which the energy spectrum is real and positive, see eq. (26), and the eigenfunctions of the Hamiltonian, see eqs. (27) and (28), are also the eigenfunctions of the operator PT , which can be seen clearly from the following equation,

$$PT \varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2) = (-1)^{n_1+n_2} \varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2), \tag{32}$$

i.e., the PT symmetry of the Hamiltonian system is unbroken.

Note that $|\epsilon| = 1$ is a critical point at which the model described by eq. (1) is no longer a free two-dimensional oscillator-like system. This can be verified after eq. (1) under this critical condition is diagonalized. When $|\epsilon| > 1$, one can see obviously from eq. (26) that the spectrum becomes complex, where the parameter ζ takes 1 or -1 . We note that the Hamiltonian depicted by eq. (1) or eq. (9) is PT symmetric, which is independent of the magnitude of the coupling constant ϵ . In the region of $|\epsilon| > 1$, eq. (27) is still the set of eigenfunctions of the Hamiltonian but no longer that of the operator PT , i.e., the PT

⁸Equivalently, it can be expressed in the notations of the PT -symmetric quantum mechanics as the PTV inner product (cf. footnote 5.):

$$\begin{aligned}
& \langle \varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2) | \varphi_{m_1 m_2}(\mathcal{X}_1, \mathcal{X}_2) \rangle_{PV} \\
& = \langle \varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2), \varphi_{m_1 m_2}(\mathcal{X}_1, \mathcal{X}_2) \rangle_{PTV} \\
& \equiv \int_{-\infty+iI_1}^{+\infty+iI_1} \int_{-\infty+iI_2}^{+\infty+iI_2} [PTV \varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2)] \varphi_{m_1 m_2}(\mathcal{X}_1, \mathcal{X}_2) d\mathcal{X}_1 d\mathcal{X}_2 \\
& = \delta_{n_1 m_1} \delta_{n_2 m_2}.
\end{aligned}$$

symmetry is broken. Let us verify this result. As either c_1 or c_2 turns out to be complex if $|\epsilon| > 1$, see eq. (29), that is, each of the two cases must happen, we take the case of a complex c_1 and a real c_2 as an example which corresponds to (i) $\zeta = 1$ and $\epsilon < -1$ or (ii) $\zeta = -1$ and $\epsilon > 1$. In this case, $\varphi_{n_1}(\mathcal{X}_1)$ is not the eigenfunction of PT although $\varphi_{n_2}(\mathcal{X}_2)$ is,

$$PT\varphi_{n_1}(\mathcal{X}_1) = (-1)^{n_1} \frac{\sqrt{\bar{c}_1}}{\sqrt[4]{\pi}} (2^{n_1} n_1!)^{-\frac{1}{2}} e^{-\frac{1}{2}(\bar{c}_1 \mathcal{X}_1)^2} H_{n_1}(\bar{c}_1 \mathcal{X}_1) \neq \text{const.} \varphi_{n_1}(\mathcal{X}_1), \quad (33)$$

where \bar{c}_1 means the complex conjugate of c_1 . The above equation gives rise to the result that $\varphi_{n_1 n_2}(\mathcal{X}_1, \mathcal{X}_2)$ is no longer the eigenfunction of PT . Thus the PT symmetry is now broken, which was ignored in ref. [14] because the eigenfunctions were not solved there.

3 Model 2: Two coupled non-Hermitian PT -symmetric Hamiltonian system

We apply the algebraic method to a more complicated model [14] which is composed of two coupled PT -symmetric Hamiltonians,

$$H = (p_1^2 + x_1^2 + i2\tau_1 x_1) + (p_2^2 + x_2^2 + i2\tau_2 x_2) + 2\epsilon x_1 x_2, \quad (34)$$

where both oscillators contain non-Hermitian terms but the interaction is Hermitian, and τ_1 and τ_2 are real parameters. We shall investigate this model by following the same way as in the above section. Here we emphasize that the analyzing procedure is almost same except for involving in more complicated calculations, thus we give prominence to the important results but omit the related computing.

First, we write the diagonalized formulation of the above Hamiltonian eq. (34),

$$H = 2a^2 \mathcal{P}_1^2 + \frac{1 + \zeta\epsilon}{2a^2} \mathcal{X}_1^2 + 2b^2 \mathcal{P}_2^2 + \frac{1 - \zeta\epsilon}{2b^2} \mathcal{X}_2^2 + \frac{\tau_1^2 + \tau_2^2 - 2\epsilon\tau_1\tau_2}{1 - \epsilon^2}, \quad (35)$$

where the variables $\mathcal{P}_1, \mathcal{P}_2$ are the same as that defined in eqs. (8) and (10), but \mathcal{X}_1 and \mathcal{X}_2 take the forms,

$$\mathcal{X}_1 = a \left(x_1 + \zeta x_2 + \frac{i(\tau_1 + \zeta\tau_2)}{1 + \zeta\epsilon} \right), \quad \mathcal{X}_2 = b \left(x_1 - \zeta x_2 + \frac{i(\tau_1 - \zeta\tau_2)}{1 - \zeta\epsilon} \right), \quad (36)$$

which are different from that of model 1 (see eq. (10)) just in the constant imaginary parts. Comparing eq. (35) with eq. (9), we see that they are almost same but have the different constant shift terms. Therefore, the important results in the two models have to have the same formulations, such as the operator V (eq. (17)), the new annihilation and creation operators (eqs. (18) and (19)), the number operator (eq. (21)), and the associated commutation relations (eqs. (20) and (22)).

Next, we give the Hamiltonian written in terms of number operators,

$$H = \sqrt{1 + \zeta\epsilon}(2N_1 + 1) + \sqrt{1 - \zeta\epsilon}(2N_2 + 1) + \frac{\tau_1^2 + \tau_2^2 - 2\epsilon\tau_1\tau_2}{1 - \epsilon^2}, \quad (37)$$

whose spectrum obviously has the form,

$$E_{n_1 n_2} = \sqrt{1 + \zeta\epsilon}(2n_1 + 1) + \sqrt{1 - \zeta\epsilon}(2n_2 + 1) + \frac{\tau_1^2 + \tau_2^2 - 2\epsilon\tau_1\tau_2}{1 - \epsilon^2}. \quad (38)$$

In the above of this section, we focus only on the case $|\epsilon| < 1$, and consequently obtain the real and positive spectrum. We note that this spectrum coincides with that given in ref. [14] where the non-Hermitian PT -symmetric Hamiltonian was dealt with by being converted to its Hermitian counterpart. Although the spectrum is the same for the two different Hamiltonians, the eigenfunctions of the non-Hermitian PT -symmetric Hamiltonian are different from that of the Hermitian counterpart. In addition, the critical point is at $|\epsilon| = 1$, and for the case $|\epsilon| > 1$, the spectrum becomes complex due to the breaking of the PT symmetry as analyzed in the subsection 2.5.

At last, we turn to the eigenfunctions of the Hamiltonian (see eq. (34), eq. (35), or eq. (37)) and the positive definiteness of their inner products, which was not studied in ref. [14]. We can work out the same eigenfunctions as eqs. (27) and (28) in which the coordinates \mathcal{X}_1 and \mathcal{X}_2 should be replaced by the ones whose operators are defined by eq. (36). As to the positive definite PV -pseudo inner product (or its equivalent PTV inner product) of the eigenfunctions, we can prove by achieving the similar calculations to that expressed by eq. (31) where the two parameters in the upper and lower limits of integration now take the values $I_1 = \frac{a(\tau_1 + \zeta\tau_2)}{1 + \zeta\epsilon}$ and $I_2 = \frac{b(\tau_1 - \zeta\tau_2)}{1 - \zeta\epsilon}$.

4 Conclusion

In this paper we apply the algebraic method to two non-Hermitian PT -symmetric quantum systems and obtain the energy spectra and eigenfunctions, and further investigate the relation between the reality of spectra and the PT symmetry of the systems. Note that $|\epsilon| = 1$ is a critical point for the two models described by eq. (1) and eq. (34). In the weak interacting region, $|\epsilon| < 1$, the spectra are real and positive and the PT symmetry is unbroken; in the strong interacting region, $|\epsilon| > 1$, the spectra are complex and the PT symmetry is broken. The spectra we obtain are exactly same as that given in ref. [14] where the eigenfunctions were circumvented because the Hamiltonians of the systems were changed. Our results show that the algebraic method is available to the non-Hermitian PT -symmetric quantum systems although it was proposed for the η_+ -pseudo Hermitian quantum systems. We prove the equivalence between the PTV inner product and the PV -pseudo inner product and confirm the positive definiteness of the inner product of eigenfunctions. In particular, due to the

reason that the operator V can be constructed in terms of Hamiltonians, our proof is free of the postulation that a Hamiltonian should possess a complete biorthonormal eigenbasis and a discrete spectrum, while such a postulation is mandatory in ref. [11] because there the related operator was constructed in terms of eigenfunctions, see also footnote 3. As a whole, we bring the non-Hermitian PT -symmetric Hamiltonian systems into the framework of the PV -pseudo Hermitian quantum mechanics and then deal with them by using the algebraic method. Comparing with the way adopted in ref. [14] that converts a non-Hermitian Hamiltonian to its Hermitian counterpart, we note that our method has the merit that keeps the Hilbert space of the non-Hermitian PT -symmetric Hamiltonian system unchanged.

We point out that the two models (see eqs. (1) and (34)) are symmetric under the transposition and therefore they are also P -pseudo Hermitian self-adjoint. However, for the models that do not have such an invariance, it is still unclear which kind of pseudo Hermitian self-adjoint symmetries they correspond to. This is an interesting problem and thus left for our further consideration in a separate work.

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